Group Structure of an Extended Lorentz Group

James Lindesay*

Abstract

In a previous paper we extended the Lorentz group to include a set of *Dirac boosts* that give a direct correspondence with a set of generators which for spin 1/2 systems are proportional to the Dirac matrices. The group is particularly useful for developing general linear wave equations beyond spin 1/2 systems. In this paper we develop explicit group properties of this extended Lorentz group to obtain group parameters that will be useful for physical calculations for systems which might manifest the group properties. This group is a subgroup of an extended Poincare group, whose structure will be developed in a subsequent paper.

*Permanent address, Department of Physics, Howard University, Washington, DC 20059

1 Introduction

When developing general, non-perturbative models for physical systems that satisfy the expected general cluster decomposition properties for a multiparticle system, one finds it particularly useful to have fields whose energy-momentum equations satisfy linear dispersions in the manner of the Dirac equation[1]. In previous papers[2][3] we have developed a set of group generators that satisfy this criterion. In particular, we demonstrated the Dirac matrices as proportional to the $\Lambda = \frac{1}{2}$ representation of the group, as well as explicitly demonstrated the $\Lambda = 1$ representation of the group.

In this paper, we will develop the group structure elements for the extended Lorentz group developed in reference[2]. In a subsequent paper, we will develop the group structure elements for the extended Poincare group developed in reference[3]. The particular representation being developed as the covering group will be seen to be a subgroup of SL(4,C). We will explicitly calculate those group structure elements that will be relavant for calculations involving systems which have the extended Lorentz group as a local gauge symmetry.

2 Group Theoretic Conventions

In order to establish our conventions, we will briefly define the group parameters that will be calculated in this paper. In terms of the group elements $\mathcal{M}$, a vector representation S satisfies

$$S(\mathcal{M}_2)\, S(\mathcal{M}_1) \;=\; S(\Phi(\mathcal{M}_2; \mathcal{M}_1)) \tag{2.1}$$

where $\Phi(\mathcal{M}_2; \mathcal{M}_1)$ is the group composition rule. The generators of infinitesimal transformations are given by

$$i\, \mathbf{X}_r \;\equiv\; \frac{\partial}{\partial \mathcal{M}^r}\, S(\mathcal{M}) \bigg|_{\mathcal{M}=\mathcal{I}}. \tag{2.2}$$

Lie structure matrices can be defined by

$$\mathbf{\Theta}_r^s(\mathcal{M}) \;\equiv\; \frac{\partial \Phi^s(\mathcal{M}'; \mathcal{M})}{\partial \mathcal{M}'^r} \bigg|_{\mathcal{M}'=\mathcal{I}} \tag{2.3}$$

from which one can obtain the group structure constants

$$c_{sn}^m \;\equiv\; \frac{\partial}{\partial \mathcal{M}^n} \mathbf{\Theta}_s^m(\mathcal{M}) \bigg|_{\mathcal{M}=\mathcal{I}} - \frac{\partial}{\partial \mathcal{M}^s} \mathbf{\Theta}_n^m(\mathcal{M}) \bigg|_{\mathcal{M}=\mathcal{I}} \tag{2.4}$$

for the Lie algebra

$$[\mathbf{X}_r, \mathbf{X}_s] \;=\; -i c_{rs}^m \mathbf{X}_m. \tag{2.5}$$

The generators transform under the representations of the group as given by the relation

$$S(\mathcal{M}^{-1})\, \mathbf{X}_r\, S(\mathcal{M}) \;=\; \mathbf{\Theta}_r^s(\mathcal{M})\, \mathbf{X}_s \tag{2.6}$$

where the matrices $\boldsymbol{\Theta}$ given by

$$\boldsymbol{\Theta}^s_r(\mathcal{M}) \;\equiv\; \frac{\partial}{\partial \mathcal{M}'^r}\Phi^s(\mathcal{M}^{-1}\,;\,\Phi(\mathcal{M}'\,;\,\mathcal{M}))\bigg|_{\mathcal{M}'=\mathcal{I}} \tag{2.7}$$

form a fundamental representation of the group.

We will ultimately want to examine the behavior of systems which have a local gauge symmetry under the group being examined. This means that we should examine derivatives of the representations. The relation

$$\boldsymbol{\Theta}^s_r(\mathcal{M})\frac{\partial}{\partial \mathcal{M}^s}S(\mathcal{M}) \;=\; i\,\mathbf{X}_r\,S(\mathcal{M}) \tag{2.8}$$

follows directly from Equations 2.1 and 2.2 This means that if we want to construct a gauge covariant derivative $\mathbf{D}_\mu \;=\; \mathbf{1}\partial_\mu - iA^r_\mu\,\mathbf{X}_r$ under the transformation S using a gauge field A^r_μ, then the gauge field will have a component a^r_μ satisfying[4][5]

$$\begin{aligned} \partial_\mu \mathcal{M}^s \;&=\; a^r_\mu(\mathcal{M})\,\boldsymbol{\Theta}^s_r(\mathcal{M}) \\ A^r_\mu(\vec{x}:\mathcal{M}) \;&=\; a^r_\mu(\mathcal{M}) + A^r_\mu(\vec{x}:\mathcal{I}). \end{aligned} \tag{2.9}$$

Under an infinitesimal gauge transformation, the gauge field then transforms in the usual manner

$$\delta A^r_\mu \;\cong\; \partial_\mu(\delta\mathcal{M}^r) \,-\, \delta\mathcal{M}^s\,c^r_{sm}\,A^m_\mu. \tag{2.10}$$

We will therefore calculate the group matrix elements $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}$ that determine the transformation properties of generators and gauge fields for this group. In subsequent papers, these elements will be used in calculations involving dynamical models for physical systems.

3 Lorentz Group Structure

The Lorentz group is a subgroup of the extended Lorentz group being developed. We will first construct the desired elements for the Lorentz group.

3.1 Pure Rotation Subgroup

The rotation subgroup representation is given by

$$\mathbf{R}(\underline{\theta}_2)\mathbf{R}(\underline{\theta}_1) \;=\; \mathbf{R}(\underline{\theta}_{(R)}(\underline{\theta}_2;\underline{\theta}_1)) \tag{3.1}$$

in terms of the group composition element $\underline{\theta}_{(R)}(\underline{\theta}_2;\underline{\theta}_1)$. The inverse element is given by $\{\underline{\theta}\}^{-1} = \{-\underline{\theta}\}$.

We can determine the group structure of the fundamental representation in a straightforward manner. For SU(2) we can explicitly write the representation in the form

$$\mathbf{R}(\underline{\theta}) \;=\; e^{i\underline{\theta}\cdot\underline{\sigma}/2} = \mathbf{1}\,cos(\frac{\theta}{2}) + i\hat{\theta}\cdot\underline{\sigma}\,sin(\frac{\theta}{2}) \tag{3.2}$$

The SU(2) composition rule satisfies

$$
\begin{aligned}
cos(\tfrac{\theta_{(R)}(\underline{\theta}_2;\underline{\theta}_1)}{2}) \;&=\; cos(\tfrac{\theta_2}{2})cos(\tfrac{\theta_1}{2}) - \hat{\theta}_2\cdot\hat{\theta}_1 sin(\tfrac{\theta_2}{2})sin(\tfrac{\theta_1}{2}) \\
\hat{\theta}_{(R)} sin(\tfrac{\theta_{(R)}(\underline{\theta}_2;\underline{\theta}_1)}{2}) \;&=\; \hat{\theta}_2 sin(\tfrac{\theta_2}{2})cos(\tfrac{\theta_1}{2}) + \hat{\theta}_1 cos(\tfrac{\theta_2}{2})sin(\tfrac{\theta_1}{2}) + \hat{\theta}_2\times\hat{\theta}_1 sin(\tfrac{\theta_2}{2})sin(\tfrac{\theta_1}{2})
\end{aligned}
\tag{3.3}
$$

By direct substitution, Equation 3.2 gives the fundamental representation:

$$
\begin{aligned}
\mathbf{\Theta}^{J_m}_{J_k}(\underline{0},\underline{\theta}) \;&=\; cos(\theta)\delta_{k,m} + (1-cos(\theta))\hat{\theta}_k\hat{\theta}_m + sin(\theta)\hat{\theta}_j\epsilon_{jkm} \\
\mathbf{\Theta}^{K_m}_{K_k}(\underline{0},\underline{\theta}) \;&=\; cos(\theta)\delta_{k,m} + (1-cos(\theta))\hat{\theta}_k\hat{\theta}_m + sin(\theta)\hat{\theta}_j\epsilon_{jkm}
\end{aligned}
\tag{3.4}
$$

The Lie structure matrices can be calculated using Equation 3.3 by examining infinitesimal $\underline{\theta}_2$;

$$
\mathbf{\Theta}^{(R)s}_r(\underline{\theta}) \;=\; \delta_{r,s} + \frac{\theta_k}{2}\epsilon_{krs} + \left(\frac{\theta}{2}cot(\frac{\theta}{2}) - 1\right)(\delta_{r,s} - \hat{\theta}_r\hat{\theta}_s) \tag{3.5}
$$

3.2 Lorentz Boosts

In general, sequential pure Lorentz boosts can be written in terms of a single pure Lorentz boost and a pure rotation:

$$
\mathbf{L}(\underline{u}_2)\,\mathbf{L}(\underline{u}_1) \;\equiv\; \mathbf{L}(\underline{u}_{(L)}(\underline{u}_2\,;\,\underline{u}_1))\,\mathbf{R}(\underline{\theta}_{(L)}(\underline{u}_2\,;\,\underline{u}_1)) \tag{3.6}
$$

The composition rule for the covering group SL(2,C) representation can be expressed

$$
\mathbf{L}_{\underline{\beta}} \;=\; e^{\underline{\beta}\cdot\underline{\sigma}/2} = \mathbf{1}\,cosh(\frac{\beta}{2}) + \hat{\beta}\cdot\underline{\sigma}\,sinh(\frac{\beta}{2}). \tag{3.7}
$$

Defining the four-velocity $\vec{u}$ using $u^0 = \sqrt{1+|\underline{u}|^2} \equiv cosh(\beta)$ and $\underline{u} \equiv \hat{\beta} sinh\beta$, the Lorentz boost takes the form

$$
\mathbf{L}(\underline{u}) \;=\; \sqrt{\frac{u^0+1}{2}}\,\mathbf{1} + \sqrt{\frac{u^0-1}{2}}\,\hat{u}\cdot\underline{\sigma} \tag{3.8}
$$

The composition rule for pure boosts satisfies

$$
\begin{aligned}
tan(\tfrac{\theta_{(L)}(\underline{u}_2;\underline{u}_1)}{2}) \;&=\; \tfrac{|\underline{u}_2\times\underline{u}_1|}{(u_2^0+1)(u_1^0+1)+\underline{u}_2\cdot\underline{u}_1} \\
\hat{\theta}_{(L)}(\underline{u}_2;\underline{u}_1) \;&=\; \tfrac{\underline{u}_2\times\underline{u}_1}{|\underline{u}_2\times\underline{u}_1|} \\
u^0_{(L)}(\underline{u}_2;\underline{u}_1) \;&=\; u_2^0\,u_1^0 \;+\; \underline{u}_2\cdot\underline{u}_1 \\
\left(\hat{u}_{(L)}(\underline{u}_2;\underline{u}_1)\,cos(\tfrac{\theta_{(L)}}{2}) - \hat{u}_{(L)}(\underline{u}_2;\underline{u}_1)\times\hat{\theta}_{(L)}\,sin(\tfrac{\theta_{(L)}}{2})\right)\sqrt{\tfrac{u^0_{(L)}-1}{2}} \;&=\; \\
\hat{u}_2\sqrt{\tfrac{u_2^0-1}{2}}\sqrt{\tfrac{u_1^0+1}{2}} + \hat{u}_1\sqrt{\tfrac{u_2^0+1}{2}}\sqrt{\tfrac{u_1^0-1}{2}}
\end{aligned}
\tag{3.9}
$$

3.3 General Lorentz Transformations

Lorentz boosts and rotations form a group of transformations whose representation will be given by the convention $\mathcal{L}(\underline{u},\underline{\theta}) \equiv \mathbf{L}(\underline{u})\mathbf{R}(\underline{\theta})$. The group composition behavior is given by

$$
\mathcal{L}(\underline{u}_2,\underline{\theta}_2)\,\mathcal{L}(\underline{u}_1,\underline{\theta}_1) \;=\; \mathbf{L}\left(\underline{u}_{(L)}(\underline{u}_2;R(\underline{\theta}_2)\underline{u}_1)\right)\,\mathbf{R}\left(\underline{\theta}_{(R)}(\underline{\theta}_{(L)}(\underline{u}_2;R(\underline{\theta}_2)\underline{u}_1);\underline{\theta}_{(R)}(\underline{\theta}_2;\underline{\theta}_1))\right). \tag{3.10}
$$

The inverse element of this representation of the Lorentz group satisfies

$$\{\underline{u}, \underline{\theta}\}^{-1} \;=\; \{-R(-\underline{\theta})\underline{u}, -\underline{\theta}\} \tag{3.11}$$

The fundamental representation matrix elements can be constructed using Equation 2.6 to obtain

$$\begin{aligned}
\mathbf{\Theta}_{J_k}^{J_m}(\underline{u},\underline{0}) \;&=\; u^0\delta_{k,m} + (1-u^0)\hat{u}_k\hat{u}_m \\
\mathbf{\Theta}_{J_k}^{K_m}(\underline{u},\underline{0}) \;&=\; u_j\epsilon_{jkm} \\
\mathbf{\Theta}_{K_k}^{J_m}(\underline{u},\underline{0}) \;&=\; -u_j\epsilon_{jkm} \\
\mathbf{\Theta}_{K_k}^{K_m}(\underline{u},\underline{0}) \;&=\; u^0\delta_{k,m} + (1-u^0)\hat{u}_k\hat{u}_m
\end{aligned} \tag{3.12}$$

The general matrix is then constructed from a rotation and sequential boost using

$$\mathbf{\Theta}_r^s(\underline{u},\underline{\theta}) \;=\; \mathbf{\Theta}_r^m(\underline{u},\underline{0})\,\mathbf{\Theta}_m^s(\underline{0},\underline{\theta}) \;=\; \mathbf{\Theta}_r^{(L)\,m}(\underline{u})\,\mathbf{\Theta}_m^{(R)\,s}(\underline{\theta}). \tag{3.13}$$

In addition, the Lie structure matrices can be shown to be given by

$$\begin{aligned}
\mathbf{\Theta}_{\theta_k}^{(\mathcal{L})\theta_j}(\underline{u},\underline{\theta}) \;&=\; \mathbf{\Theta}_k^{(R)j}(\underline{\theta}) \\
\mathbf{\Theta}_{u_k}^{(\mathcal{L})\theta_j}(\underline{u},\underline{\theta}) \;&=\; \mathbf{\Theta}_{u_k}^{(L)\theta_m}(\underline{u})\mathbf{\Theta}_m^{(R)j}(\underline{\theta}) \\
\mathbf{\Theta}_{u_k}^{(L)\theta_j}(\underline{u}) \;&=\; -\tfrac{u_m}{u^0+1}\,\epsilon_{mkj} \\
\mathbf{\Theta}_{\theta_k}^{(\mathcal{L})u_j}(\underline{u},\underline{\theta}) \;&=\; \mathbf{\Theta}_{\theta_k}^{(L)u_j}(\underline{u}) \;=\; u_m\,\epsilon_{mkj} \\
\mathbf{\Theta}_{u_k}^{(\mathcal{L})u_j}(\underline{u},\underline{\theta}) \;&=\; \mathbf{\Theta}_{u_k}^{(L)u_j}(\underline{u}) \;=\; u^0\,\delta_{k,j}
\end{aligned} \tag{3.14}$$

To complete this section, we will establish our convention for the Lorentz transformation matrices on four-vectors. Define $\mathcal{R}_\mu^\nu$ and $\mathcal{L}_\mu^\nu$ which act on (covariant) 4-vectors according to $\Lambda_\mu^\nu\omega_\nu = \omega'_\mu$

$$\begin{aligned}
\mathcal{R}_k^m(\underline{\theta}) \;&=\; cos(\theta)\delta_{k,m} + (1-cos(\theta))\hat{\theta}_k\hat{\theta}_m + sin(\theta)\hat{\theta}_j\epsilon_{jkm} \\
\mathcal{R}_0^0(\underline{\theta}) \;&=\; 1 \\
\\
\mathcal{L}_k^m(\underline{u}) \;&=\; \delta_{k,m} - (1-u^0)\hat{u}_k\hat{u}_m \\
\mathcal{L}_0^m(\underline{u}) \;&=\; -u_m \;=\; \mathcal{L}_m^0(\underline{u}) \\
\mathcal{L}_0^0(\underline{u}) \;&=\; u^0
\end{aligned} \tag{3.15}$$

The form of the infinitesimal 4-Lorentz generators

$$\begin{aligned}
(\mathcal{J}_m)_\mu^{\;\nu} \;&\equiv\; \tfrac{\partial}{\partial\theta_m}\mathcal{R}_\mu^\nu(\underline{\theta})\Big|_{\underline{\theta}=\underline{0}} \\
(\mathcal{K}_m)_\mu^{\;\nu} \;&\equiv\; \tfrac{\partial}{\partial u_m}\mathcal{L}_\mu^\nu(\underline{u})\Big|_{\underline{u}=\underline{0}}
\end{aligned} \tag{3.16}$$

has non-vanishing elements given by

$$
\begin{aligned}
(\mathcal{J}_m)_j^{\,k} &= \epsilon_{mjk} \\
(\mathcal{K}_m)_0^{\,k} &= -\delta_{m,k} = (\mathcal{K}_m)_k^{\,0}\,.
\end{aligned}
\tag{3.17}
$$

4 Extended Lorentz Group Structure

The primary purpose of this paper is the development of the group structure for the extended Lorentz group developed in a previous paper[2]. The covering group will be developed as a subgroup of SL(4,C).

4.1 Form of $\Lambda = \frac{1}{2}$ Matrices

The forms of the lowest dimensional matrices of the extended Lorentz group corresponding to $\Lambda = \frac{1}{2}$ can be expressed in terms of the Pauli spin matrices as shown below:

$$
\begin{aligned}
\mathbf{\Gamma^0} &= \tfrac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \equiv \tfrac{1}{2}\gamma^{\mathbf{0}} \qquad & \underline{\mathbf{J}} &= \tfrac{1}{2}\begin{pmatrix} \underline{\sigma} & \mathbf{0} \\ \mathbf{0} & \underline{\sigma} \end{pmatrix} \\
\underline{\mathbf{\Gamma}} &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \underline{\sigma} \\ -\underline{\sigma} & \mathbf{0} \end{pmatrix} \equiv \tfrac{1}{2}\underline{\gamma} \qquad & \underline{\mathbf{K}} &= -\tfrac{i}{2}\begin{pmatrix} \mathbf{0} & \underline{\sigma} \\ \underline{\sigma} & \mathbf{0} \end{pmatrix}
\end{aligned}
\tag{4.1}
$$

The Γ^μ matrices can directly be seen to be proportional to a representation of the Dirac matrices[1][6] .

4.2 Dirac Boosts

Group transformations parameterized by elements $\vec{\omega}$ conjugate to the generators Γ^μ will be referred to as pure Dirac boosts. In general, sequential pure Dirac boosts can be written in terms of a single pure Dirac boost, a pure Lorentz boost, and a pure rotation:

$$
\mathbf{W}(\vec{\omega}_2)\,\mathbf{W}(\vec{\omega}_1) \;\equiv\; \mathbf{W}(\vec{\omega}_{(D)}(\vec{\omega}_2\,;\vec{\omega}_1))\,\mathbf{L}(\underline{u}_{(D)}(\vec{\omega}_2\,;\vec{\omega}_1))\,\mathbf{R}(\underline{\theta}_{(D)}(\vec{\omega}_2\,;\vec{\omega}_1)). \tag{4.2}
$$

To develop the composition rules for the lowest dimensional representation, the four-vector magnitude and direction will be expressed

$$
\begin{aligned}
\omega &\equiv \sqrt{-\vec{\omega}\cdot\vec{\omega}} \\
\vec{\omega} &= \omega\,\vec{q}.
\end{aligned}
\tag{4.3}
$$

This allows the Dirac representation to be expressed in terms of the Dirac matrices in the form

$$
e^{i\vec{\omega}\cdot\vec{\gamma}/2} = \begin{cases} \mathbf{1}cosh(\frac{\omega}{2}) + i\vec{q}\cdot\vec{\gamma}sinh(\frac{\omega}{2}) & \vec{q}\cdot\vec{q} = +1 \\ \mathbf{1} + i\vec{\omega}\cdot\vec{\gamma}/2 & \vec{\omega}\cdot\vec{\omega} = 0 \\ \mathbf{1}cos(\frac{\omega}{2}) + i\vec{q}\cdot\vec{\gamma}sin(\frac{\omega}{2}) & \vec{q}\cdot\vec{q} = -1 \end{cases} \tag{4.4}
$$

For this representation, the composition elements satisfy

$$
\begin{array}{l}
cos(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}+1}{2}}\,cos(\frac{\theta_{(D)}}{2}) \;=\; cos(\frac{\omega_2}{2})\,cos(\frac{\omega_1}{2}) \;+\; \vec{q}_2\cdot\vec{q}_1\,sin(\frac{\omega_2}{2})\,sin(\frac{\omega_1}{2}),\\
cos(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}+1}{2}}\,sin(\frac{\theta_{(D)}}{2})\,\hat{\theta}_{(D)} \;=\; sin(\frac{\omega_2}{2})\,sin(\frac{\omega_1}{2})\,\underline{q}_2\times\underline{q}_1,\\
cos(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}-1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,\hat{u}_{(D)}+cos(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}-1}{2}}\,sin(\frac{\theta_{(D)}}{2})\,\hat{\theta}_{(D)}\times\hat{u}_{(D)} \;=\\
\qquad sin(\frac{\omega_2}{2})\,sin(\frac{\omega_1}{2})\,(\underline{q}_2 q_{1\,0}-\underline{q}_1 q_{2\,0}),\\
sin(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}+1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,q_{(D)\,0}+sin(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}-1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,\underline{q}_{(D)}\cdot\hat{u}_{(D)}+\\
-sin(\frac{\omega_{(D)}}{2})\sqrt{\frac{u^0_{(D)}-1}{2}}sin(\frac{\theta_{(D)}}{2})(\hat{\theta}_{(D)}\times\underline{q}_{(D)})\cdot\hat{u}_{(D)}=q_{2\,0}sin(\frac{\omega_2}{2})cos(\frac{\omega_1}{2})+q_{1\,0}cos(\frac{\omega_2}{2})sin(\frac{\omega_1}{2}),\\
sin(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}+1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,\underline{q}_{(D)}+sin(\frac{\omega_{(D)}}{2})\,\sqrt{\frac{u^0_{(D)}-1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,q_{(D)\,0}\hat{u}_{(D)}+\\
sin(\frac{\omega_{(D)}}{2})\sqrt{\frac{u^0_{(D)}+1}{2}}sin(\frac{\theta_{(D)}}{2})(\hat{\theta}_{(D)}\times\underline{q}_{(D)})+sin(\frac{\omega_{(D)}}{2})\sqrt{\frac{u^0_{(D)}-1}{2}}sin(\frac{\theta_{(D)}}{2})\,q_{(D)\,0}\,\hat{\theta}_{(D)}\times\hat{u}_{(D)} \;=\\
\qquad \underline{q}_2 sin(\frac{\omega_2}{2})cos(\frac{\omega_1}{2})+\underline{q}_1 cos(\frac{\omega_2}{2})sin(\frac{\omega_1}{2})
\end{array}
\tag{4.5}
$$

with constraints given by

$$
\begin{array}{l}
\hat{u}_{(D)}\cdot\hat{\theta}_{(D)} \;=\; 0\\
\underline{q}_{(D)}\cdot\hat{\theta}_{(D)} \;=\; 0\\
\sqrt{\frac{u^0_{(D)}-1}{2}}\,cos(\frac{\theta_{(D)}}{2})\,\underline{q}_{(D)}\times\hat{u}_{(D)}+\\
\sqrt{\frac{u^0_{(D)}-1}{2}}\,sin(\frac{\theta_{(D)}}{2})\,\underline{q}_{(D)}\cdot\hat{u}_{(D)}\,\hat{\theta}_{(D)}+\sqrt{\frac{u^0_{(D)}+1}{2}}\,sin(\frac{\theta_{(D)}}{2})\,q_{(D)\,0}\,\hat{\theta}_{(D)} \;=\; 0
\end{array}
\tag{4.6}
$$

As was done for the Lorentz group, the fundamental representation matrix elements can be constructed

using Equation 2.6 to obtain

$$
\begin{aligned}
\mathbf{\Theta}_{J_k}^{J_s}(\vec{\omega},\underline{0},\underline{0}) &= \left(1+|\underline{q}|^2(1-cos(\omega))\right)\delta_{k,s} - q_k q_s(1-cos(\omega)) \\
\mathbf{\Theta}_{J_k}^{K_s}(\vec{\omega},\underline{0},\underline{0}) &= q_0 q_j(1-cos(\omega))\epsilon_{jks} \\
\mathbf{\Theta}_{J_k}^{\Gamma^0}(\vec{\omega},\underline{0},\underline{0}) &= 0 \\
\mathbf{\Theta}_{J_k}^{\Gamma^s}(\vec{\omega},\underline{0},\underline{0}) &= q_j sin(\omega)\epsilon_{jks} \\
\mathbf{\Theta}_{K_k}^{J_s}(\vec{\omega},\underline{0},\underline{0}) &= q_0 q_j(1-cos(\omega))\epsilon_{jks} \\
\mathbf{\Theta}_{K_k}^{K_s}(\vec{\omega},\underline{0},\underline{0}) &= \left(cos(\omega)-|\underline{q}|^2(1-cos(\omega))\right)\delta_{k,s} + q_k q_s(1-cos(\omega)) \\
\mathbf{\Theta}_{K_k}^{\Gamma^0}(\vec{\omega},\underline{0},\underline{0}) &= -q_k sin(\omega) \\
\mathbf{\Theta}_{K_k}^{\Gamma^s}(\vec{\omega},\underline{0},\underline{0}) &= -q_0 sin(\omega)\delta_{k,s} \\
\mathbf{\Theta}_{\Gamma^0}^{J_s}(\vec{\omega},\underline{0},\underline{0}) &= 0 \\
\mathbf{\Theta}_{\Gamma^0}^{K_s}(\vec{\omega},\underline{0},\underline{0}) &= -q_s sin(\omega) \\
\mathbf{\Theta}_{\Gamma^0}^{\Gamma^0}(\vec{\omega},\underline{0},\underline{0}) &= \left(1+|\underline{q}|^2(1-cos(\omega))\right) \\
\mathbf{\Theta}_{\Gamma^0}^{\Gamma^s}(\vec{\omega},\underline{0},\underline{0}) &= q_0 q_s(1-cos(\omega)) \\
\mathbf{\Theta}_{\Gamma^k}^{J_s}(\vec{\omega},\underline{0},\underline{0}) &= -q_j sin(\omega)\epsilon_{jks} \\
\mathbf{\Theta}_{\Gamma^k}^{K_s}(\vec{\omega},\underline{0},\underline{0}) &= q_0 sin(\omega)\delta_{k,s} \\
\mathbf{\Theta}_{\Gamma^k}^{\Gamma^0}(\vec{\omega},\underline{0},\underline{0}) &= -q_0 q_s(1-cos(\omega)) \\
\mathbf{\Theta}_{\Gamma^k}^{\Gamma^s}(\vec{\omega},\underline{0},\underline{0}) &= cos(\omega))\delta_{k,s} - q_k q_s(1-cos(\omega))
\end{aligned}
\tag{4.7}
$$

The non-vanishing Lie structure matrices are given by

$$
\begin{aligned}
\mathbf{\Theta}_{\omega_k}^{(D)\theta_j}(\vec{\omega}) &= \epsilon_{jkm}\, q_m\, tan(\tfrac{\omega}{2}) \\
\mathbf{\Theta}_{\omega_0}^{(D)u_j}(\vec{\omega}) &= -q_j\, tan(\tfrac{\omega}{2}) \\
\mathbf{\Theta}_{\omega_k}^{(D)u_j}(\vec{\omega}) &= \delta_{j,k}\, q_0\, tan(\tfrac{\omega}{2}) \\
\mathbf{\Theta}_{\omega_\mu}^{(D)\omega_\beta}(\vec{\omega}) &= \tfrac{\omega}{2}\, cot(\tfrac{\omega}{2})\, \delta_\beta^\mu \;-\; \left(1-\tfrac{\omega}{2} cot(\tfrac{\omega}{2})\right)\, q_\beta\, q^\mu
\end{aligned}
\tag{4.8}
$$

4.3 Lorentz Transformations on Dirac Boosts

We next examine the effect of Lorentz transformations on the Dirac boosts. Utilizing the Equation 2.6 for this representation, we can read off various elements of the fundamental representation. For pure

rotations

$$
\begin{aligned}
\boldsymbol{\Theta}_{J_k}^{J_m}(\vec{0},\underline{0},\underline{\theta}) &= cos(\theta)\delta_{k,m} + (1-cos(\theta))\hat{\theta}_k\hat{\theta}_m + sin(\theta)\hat{\theta}_j\epsilon_{jkm} \\
\boldsymbol{\Theta}_{K_k}^{K_m}(\vec{0},\underline{0},\underline{\theta}) &= cos(\theta)\delta_{k,m} + (1-cos(\theta))\hat{\theta}_k\hat{\theta}_m + sin(\theta)\hat{\theta}_j\epsilon_{jkm} \\
\boldsymbol{\Theta}_{\Gamma^\mu}^{\Gamma^\nu}(\vec{0},\underline{0},\underline{\theta}) &= \mathcal{R}_\mu^\nu(\underline{\theta})
\end{aligned}
\tag{4.9}
$$

and for pure Lorentz boosts

$$
\begin{aligned}
\boldsymbol{\Theta}_{J_k}^{J_m}(\vec{0},\underline{u},\underline{0}) &= u^0\delta_{k,m} + (1-u^0)\hat{u}_k\hat{u}_m \\
\boldsymbol{\Theta}_{J_k}^{K_m}(\vec{0},\underline{u},\underline{0}) &= -u_j\epsilon_{kjm} \\
\boldsymbol{\Theta}_{K_k}^{J_m}(\vec{0},\underline{u},\underline{0}) &= u_j\epsilon_{kjm} \qquad . \\
\boldsymbol{\Theta}_{K_k}^{K_m}(\vec{0},\underline{u},\underline{0}) &= u^0\delta_{k,m} + (1-u^0)\hat{u}_k\hat{u}_m \\
\boldsymbol{\Theta}_{\Gamma^\mu}^{\Gamma^\nu}(\vec{0},\underline{u},\underline{0}) &= \mathcal{L}_\mu^\nu(\underline{u})
\end{aligned}
\tag{4.10}
$$

The general fundamental transformation matrix is then given by

$$
\boldsymbol{\Theta}_r^s(\vec{\omega},\underline{u},\underline{\theta}) = \boldsymbol{\Theta}_r^n(\vec{\omega},\underline{0},\underline{0})\,\boldsymbol{\Theta}_n^m(\vec{0},\underline{u},\underline{0})\,\boldsymbol{\Theta}_m^s(\vec{0},\underline{0},\underline{\theta}). \tag{4.11}
$$

4.4 Extended Group Transformations

To complete our description of the group structure of this extended Lorentz group, we will explicitly demonstrate the group composition rule and Lie structure elements of the complete group. The representation we develop will be constructed by sequential pure rotation, Lorentz boost, and Dirac boost:

$$
\mathbf{M}(\vec{\omega},\underline{u},\underline{\theta}) \equiv \mathbf{W}(\vec{\omega})\,\mathbf{L}(\underline{u})\,\mathbf{R}(\underline{\theta}) \tag{4.12}
$$

The group composition rule then defines elements in this same manner

$$
\begin{aligned}
&\mathbf{M}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2)\,\mathbf{M}(\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1) \equiv \\
&\mathbf{W}(\vec{\omega}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1))\,\mathbf{L}(\underline{u}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1))\,\mathbf{R}(\underline{\theta}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1))
\end{aligned}
\tag{4.13}
$$

The inverse element can be shown to be given by

$$
\{\vec{\omega},\underline{u},\underline{\theta}\}^{-1} = \{-\mathcal{R}(-\underline{\theta})\,\mathcal{L}(-\underline{u})\,\vec{\omega}, -R(-\underline{\theta})\underline{u}, -\underline{\theta}\} \tag{4.14}
$$

where $(\Lambda\vec{\omega})_\mu = \Lambda_\mu^\nu\omega_\nu$. The group composition elements can be expressed using previously constructed functions in terms of pure boosts and rotations:

$$
\begin{aligned}
&\vec{\omega}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1) = \vec{\omega}_{(D)}\,(\vec{\omega}_2\,;\,\mathcal{L}(\underline{u}_2)\mathcal{R}(\underline{\theta}_2)\vec{\omega}_1) \\
&\underline{u}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1) = \\
&\underline{u}_{(L)}\left(\underline{u}_{(D)}\,(\vec{\omega}_2\,;\,\mathcal{L}(\underline{u}_2)\mathcal{R}(\underline{\theta}_2)\vec{\omega}_1)\,;\,\mathcal{R}(\underline{\theta}_{(D)}\,(\vec{\omega}_2\,;\,\mathcal{L}(\underline{u}_2)\mathcal{R}(\underline{\theta}_2)\vec{\omega}_1))\,\underline{u}_{(L)}\,(\underline{u}_2\,;\,\mathcal{R}(\underline{\theta}_2)\underline{u}_1)\right) \\
&\underline{\theta}(\vec{\omega}_2,\underline{u}_2,\underline{\theta}_2\,;\,\vec{\omega}_1,\underline{u}_1,\underline{\theta}_1) = \\
&\underline{\theta}_{(R)}\left(\underline{\theta}_{(D)}\,(\vec{\omega}_2\,;\,\mathcal{L}(\underline{u}_2)\mathcal{R}(\underline{\theta}_2)\vec{\omega}_1)\,;\,\underline{\theta}_{(R)}\left(\underline{\theta}_{(L)}\,(\underline{u}_2\,;\,\mathcal{R}(\underline{\theta}_2)\underline{u}_1)\,;\,\underline{\theta}_{(R)}(\underline{\theta}_2\,;\,\underline{\theta}_1)\right)\right)
\end{aligned}
\tag{4.15}
$$

4.5 Lie Structure Elements

Finally, we can use the composition rules given in Equation 4.15, along with the definition given in Equation 2.3, to explicitly develop the Lie structure matrices

$$
\begin{aligned}
\mathbf{\Theta}_{\omega_\nu}^{\omega_\mu}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_{\omega_\nu}^{(D)\,\omega_\mu}(\vec{\omega}) \\
\mathbf{\Theta}_{u_j}^{\omega_\mu}(\vec{\omega},\underline{u},\underline{\theta}) &= (\mathcal{K}_j)_\mu^\beta\,\omega_\beta \\
\mathbf{\Theta}_{\theta_j}^{\omega_\mu}(\vec{\omega},\underline{u},\underline{\theta}) &= (\mathcal{J}_j)_\mu^\beta\,\omega_\beta \\
\mathbf{\Theta}_{\omega_\nu}^{u_k}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_{\omega_\nu}^{(D)\,u_m}(\vec{\omega})\,\mathbf{\Theta}_{u_m}^{(L)\,u_k}(\underline{u}) + \mathbf{\Theta}_{\omega_\nu}^{(D)\,\theta_m}(\vec{\omega})\;(\mathcal{J}_m)_k^s\;u_s \\
\mathbf{\Theta}_{u_j}^{u_k}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_{u_j}^{(L)\,u_k}(\underline{u}) \\
\mathbf{\Theta}_{\theta_j}^{u_k}(\vec{\omega},\underline{u},\underline{\theta}) &= (\mathcal{J}_j)_k^s\;u_s \\
\mathbf{\Theta}_{\omega_\nu}^{\theta_k}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_{\omega_\nu}^{(D)\,\theta_m}(\vec{\omega})\,\mathbf{\Theta}_m^{(R)\,k}(\underline{\theta}) \\
\mathbf{\Theta}_{u_j}^{\theta_k}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_{u_j}^{(L)\,\theta_m}(\underline{u})\,\mathbf{\Theta}_m^{(R)\,k}(\underline{\theta}) \\
\mathbf{\Theta}_{\theta_j}^{\theta_k}(\vec{\omega},\underline{u},\underline{\theta}) &= \mathbf{\Theta}_j^{(R)\,k}(\underline{\theta})
\end{aligned}
\tag{4.16}
$$

We will be able to utilize these matrices in general gauge transformations for systems which have a local gauge symmetry in this group.

5 Conclusions

We have demonstrated an explicit group representation of the extended Lorentz group developed in reference[2]. The fundamental representation matrices can be seen to indeed form a group of transformations. Lie structure matrices that are needed for the homogeneous part of transformation of gauge fields have also be explicitly calculated. Further group properties, such as vector recoupling coefficients to express combined systems in terms of combinations of irreducible components, will be left for subsequent study.

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References

[1] P.A.M. Dirac, Proc. Roy. Soc. (London), A117, 610 (1928); ibid, A118, 351 (1928)

[2] J. Lindesay, Linear Spinor Field Equations for Arbitrary Spins, math-ph/0308003 (2003)

[3] J. Lindesay, An Extended Poincare Algebra for Linear Spinor Field Equations, math-ph/0308015 (2003)

[4] J.V. Lindesay and H.L. Morrison, The Geometry of Quantum Flow, in *Mathematical Analysis of Physical Systems*, R.E. Mickens, ed., Van Nostrand Reinhold, New York, p 135 (1985)

[5] H.L. Morrison and J.V. Lindesay, Galilean Presymmetry and the Quantization of Circulation, Journal of Low Temperature Physics 26, 899-907 (1977)

[6] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, p17